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Existence of stationary solutions to the Vlasov–Poisson–Boltzmann system

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Abstract

In this paper, we study the existence of stationary solutions to the Vlasov–Poisson–Boltzmann system when the background density function tends to a positive constant with a very mild decay rate as $|x| \to \infty$. In fact, the stationary Vlasov–Poisson–Boltzmann system can be written into an elliptic equation with exponential nonlinearity. Under the assumption on the decay rate being $(\ln(e+|x|))^{-\alpha}$ for some $\alpha > 0$, it is shown that this elliptic equation has a unique solution. This result generalizes the previous work [R. Glassey, J. Schaeffer, Y. Zheng, Steady states of the Vlasov–Poisson–Fokker–Planck system, J. Math. Anal. Appl. 202 (1996) 1058–1075] where the decay rate $(1+|x|)^{-1/2}$ is assumed.

Keywords: Vlasov–Poisson–Boltzmann system; Stationary solutions; Nonlinear elliptic equation

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1. Introduction

The Vlasov–Poisson–Boltzmann system which describes the charged dilute particles (e.g., electron) in the absence of a magnetic field can be written as

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \nabla_x \phi \cdot \nabla_v f = Q(f, f), \\ \Delta \phi = \int\limits_{\mathbf{R}^3} f \, dv - \rho_0(x). \end{cases} \tag{1.1}$$

Here $f = f(x, v, t)$ is the distribution function for particles at time $t \geqslant 0$ and spatial position $x = (x_1, x_2, x_3) \in \mathbf{R}^3$ with velocity $v = (v_1, v_2, v_3) \in \mathbf{R}^3$. $\phi = \phi(x, t)$ is the self-consistent electric potential governed by the Poisson equation. Here, $\rho_0(x) > 0$ is the given background density function. As usual, the Boltzmann collision operator $Q(f, f)$ takes the form

$$Q(f, f)(v) = \int\limits_{\mathbf{R}^3 \times S^2} \{ f(v') f(v'_*) - f(v) f(v_*) \} B(|v - v_*|, \theta) \, dv \, d\omega, \tag{1.2}$$

where

$$\cos\theta = \frac{(v - v_*) \cdot \omega}{|v - v_*|},$$

and

$$v' = v - [(v - v_*) \cdot \omega]\omega, \quad v'_* = v_* + [(v - v_*) \cdot \omega]\omega, \quad \omega \in S^2,$$

which are the relations between velocities before and after the elastic collision. That is:

$$v + v_* = v' + v'_*, \qquad |v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2.$$

In this paper, we are going to prove the existence of stationary solutions to the Vlasov–Poisson–Boltzmann system (1.1). Thus, similar to [4,7,8], we look for steady solutions in the form of

$$\big(f(x, v, t), \phi(x, t)\big) = \big(\exp(\Phi(x)) M[\bar\rho, 0, \bar\theta](v), \ R\bar\theta \Phi(x)\big). \tag{1.3}$$

Here, $\Phi(x)$ is to be determined and $M(v)$ is a global Maxwellian distribution with zero bulk velocity, i.e.,

$$M(v) = M[\bar\rho, 0, \bar\theta](v) \equiv \frac{\bar\rho}{(2\pi R\bar\theta)^{3/2}} \exp\left(-\frac{|v|^2}{2R\bar\theta}\right), \tag{1.4}$$

where $\bar\rho$, $\bar\theta$ and R are positive constants. Without loss of generality, we let $R = 1$. By substituting (1.3) and (1.4) into (1.1) and using the basic property

$$Q(M, M)(v) = 0,$$

we have the following nonlinear elliptic equation for $\Phi(x)$:

$$-\bar\theta \Delta \Phi + \bar\rho \exp(\Phi) = \rho_0. \tag{1.5}$$

In the following discussion, we assume that $\Phi(x)$ vanishes at infinity, i.e.,

$$\Phi(x) \to 0, \quad \text{as } |x| \to \infty. \tag{1.6}$$

Under this formulation, the existence of a smooth function $\Phi(x)$ to (1.5) and (1.6) is equivalent to the existence of a stationary solution with zero bulk velocity to the Vlasov–Poisson–Boltzmann system. Thus, we will concentrate on the problem (1.5) and (1.6) in the following discussion.

For the existence of solutions to the problem (1.5) and (1.6), the previous work in [9] gave an affirmative answer when the background density $\rho_0(x)$ tends to a positive constant state $\bar{\rho}$ with the decay rate $(1+|x|)^{-1/2}$. The main result in this paper is to show that under a weaker decay rate assumption, the existence and the uniqueness of the solution to the problem (1.5) and (1.6) still hold. In fact, as long as $\rho_0(x)$ has a decay rate as $(\ln(e+|x|))^{-\alpha}$, where α is some positive constant, the elliptic equation (1.5) has a solution satisfying (1.6).

In what follows, for simplicity, we suppose that $M(v)$ is the normalized global Maxwellian distribution:

$$M(v) = M[1,0,1](v) \equiv \frac{1}{(2\pi)^{3/2}} \exp\left(-\frac{|v|^2}{2}\right), \tag{1.7}$$

so that the problem (1.5) with (1.6) becomes

$$\begin{cases} -\Delta\Phi + \exp(\Phi) = \rho_0, \\ \Phi(x) \to 0, \quad \text{as } |x| \to \infty. \end{cases} \tag{1.8}$$

The main result in this paper can be stated as follows.

Theorem 1.1. *Suppose that $\rho_0(x) \in L^\infty(\mathbf{R}^3)$ satisfies*

$$\big|\rho_0(x) - 1\big| \leqslant C\big(\ln\big(e+|x|\big)\big)^{-\alpha}, \tag{1.9}$$

where C and α are some positive constants. The problem (1.8) *has a unique smooth solution $\Phi(x) \in C^\infty(\mathbf{R}^3)$ satisfying*

$$-D_2\big(\ln\big(e+|x|\big)\big)^{-\alpha} \leqslant \Phi(x) \leqslant D_1\big(\ln\big(e+|x|\big)\big)^{-\alpha}, \tag{1.10}$$

where D_1 and D_2 are two positive constants depending on C and α. That is, the Vlasov–Poisson–Boltzmann system has the stationary solution

$$(f,\phi) = \big(\exp\big(\Phi(x)\big)M[1,0,1](v), \Phi(x)\big). \tag{1.11}$$

There are a lot of studies on the mathematical theory of the Vlasov–Poisson–Boltzmann system. For example, when there is no background charge, i.e., $\rho_0(x) \equiv 0$, the large time behavior of weak solutions was studied in [5]. Global existence of DiPerna–Lions renormalized solutions with arbitrary amplitude to the initial boundary value problem was obtained in [11], and the global existence of classical solutions with small amplitude for the Cauchy problem was given in [10]. When the background density $\rho_0(x)$ is a positive constant, the existence of global solutions around a global Maxwellian distribution function was proved in [12,13]. See also [2,3,6] for the related topics on the Vlasov–Poisson–Fokker–Planck system.

For more general background charge $\rho_0(x)$, the authors in [1] studied this system in a bounded domain with the incoming boundary condition and proved the convergence of the solution to a unique stationary solution as the time goes to infinity by using the relative entropy method. We expect that the result in this paper can be used to investigate the stability of stationary solutions in the general setting.

The rest of the paper is arranged as follows. In the next section, we will present the main ideas in proving Theorem 1.1 by showing some preliminary lemmas. The existence and uniqueness of the solution to the problem (1.8) will be given in the last section.

2. Preliminaries

In this section, we first reformulate the problem (1.8) and then give some basic lemmas for later use. For this, set $u(x) = -\Phi(x)$ and $\rho_0(x) = 1 + \eta(x)$. Then the problem (1.8) becomes

$$\begin{cases} \Delta u + \exp(-u) = 1 + \eta, & x \in \mathbf{R}^3, \\ u(x) \to 0, & \text{as } |x| \to \infty. \end{cases} \tag{2.1}$$

Define a function $g : \mathbf{R} \to \mathbf{R}$, by

$$g(y) = \exp(-y) + A^2 y - 1, \tag{2.2}$$

where the positive constant A will be chosen later. To apply the method of lower and upper solutions, we rewrite (2.1) into a linear equation by considering a solution operator $\mathcal{L}$ satisfying

$$\begin{cases} \left(\Delta - A^2\right)\mathcal{L}u = \eta - g(u), & x \in R^3, \\ \mathcal{L}u(x) \to 0, & \text{as } |x| \to \infty. \end{cases} \tag{2.3}$$

Here, u is viewed as a given function in the equation for $\mathcal{L}u$. The rest of the paper is to show that under some conditions, there exists a function $\bar{U}$ such that $\mathcal{L}\bar{U} = \bar{U}$.

With these notations, it is straightforward to prove the following lemma. We omit its proof for brevity.

Lemma 2.1. *The function g and the operator $\mathcal{L}$ have the following properties.*

(1) *g is a strictly increasing smooth function on $[-\ln A^2, \infty)$ so that the inverse function $g^{-1} : [g_{\min}, \infty) \to [-\ln A^2, \infty)$ is well defined. Here,*

$$g_{\min} = g\left(-\ln A^2\right) = -A^2\left(\ln A^2 - 1\right) - 1.$$

(2) *For any $x \in \mathbf{R}$,*

$$g''(x) = \exp(-x) \geqslant 0 \quad \text{and} \quad g(x) \geqslant g'(0)x = \left(A^2 - 1\right)x.$$

(3) *$\mathcal{L}$ has the explicit expression*:

$$\mathcal{L}u = -\frac{1}{4\pi|x|}\exp\left(-A|x|\right) * \left(\eta - g(u)\right). \tag{2.4}$$

Furthermore, $\mathcal{L}$ is a monotone operator in the sense that if $-\ln A^2 \leqslant U \leqslant W$ with $U, W \to 0$ as $|x| \to \infty$, then $\mathcal{L}U \leqslant \mathcal{L}W$.

Notice that if u is a fixed point of $\mathcal{L}$ defined in (2.3), then u is a smooth solution to the problem (2.1). Here the smoothness of u comes directly from the elliptic property of $\mathcal{L}$ as long as $u \in L^1_{\mathrm{loc}}(\mathbf{R}^3)$. Therefore, by Lemma 2.1, the solution to the problem (2.1) can be obtained by using the following iteration process similar to [9]. Suppose that there exist functions U and W satisfying

$$-\ln A^2 \leqslant U \leqslant W, \quad U, W \to 0 \quad \text{as } |x| \to \infty, \tag{2.5}$$

and

$$U \leqslant \mathcal{L}U, \qquad \mathcal{L}W \leqslant W. \tag{2.6}$$

Then we have

$$U \leqslant \mathcal{L}U \leqslant \cdots \leqslant \mathcal{L}^n U \leqslant \mathcal{L}^n W \leqslant \cdots \leqslant \mathcal{L}W \leqslant W$$

for all positive integers n. By the monotone convergence theorem, $\mathcal{L}^n U$ converges to some function $\bar{U}$ as n tends to ∞. Thus, the dominated convergence theorem yields

$$\bar{U} \leftarrow \mathcal{L}^{n+1} U = \mathcal{L}\big(\mathcal{L}^n U\big) \to \mathcal{L}\bar{U},$$

which implies that $\bar{U}$ is a solution to (2.1). Hence, it remains only to show that there are functions U and W satisfying (2.5) and (2.6). For this, we give the following definition.

Definition 2.2. The functions U and W satisfying conditions (2.5) and (2.6) are called a subsolution and a supersolution to the problem (2.1), respectively.

The construction of a subsolution U and a supersolution W to the nonlinear elliptic equation (2.1) is left to the next section. Here, we present a lemma on the property of the operator $(\Delta - A^2)$.

Denote

$$\sigma = 1 + \ln\big(1 + r^2\big), \tag{2.7}$$

with

$$r = |x| = \left(\sum_{i=1}^{3} x_i^2\right)^{1/2}, \quad x = (x_1, x_2, x_3) \in \mathbf{R}^3. \tag{2.8}$$

Since the positive parameter α in the decay rate $(\ln(e + |x|))^{-\alpha}$ can be arbitrary, it is equivalent to use the decay rate in the form of σ^{-B} in the following discussion. Here, B is a positive constant. Notice that the function σ^{-B} is differentiable at $x = 0$ with its first order derivatives equal to zero.

Lemma 2.3. *For any constant $B > 0$, we have*

$$\big(\Delta - A^2\big)\big(\sigma^{-B}\big) = -\big(A^2 - a(r)\big)\sigma^{-B}, \tag{2.9}$$

where

$$a(r) = \frac{4B(B+1)r^2}{(1+r^2)^2}\sigma^{-2} + \frac{4Br^2}{(1+r^2)^2}\sigma^{-1} - \frac{6B}{1+r^2}\sigma^{-1}. \tag{2.10}$$

In particular, when $0 < B < 1/2$, we have

$$-6B \leqslant a(r) \leqslant -\frac{6B\sigma^{-2}}{(1+r^2)^2} < 0. \tag{2.11}$$

Proof. For any constant $B > 0$, direct calculation yields

$$\big(\Delta - A^2\big)\big(\sigma^{-B}\big) = \frac{d^2}{dr^2}\big(\sigma^{-B}\big) + \frac{2}{r}\frac{d\sigma}{dr} - A^2\sigma^{-B} = -\big(A^2 - a(r)\big)\sigma^{-B},$$

where $a(r)$ is given by (2.10). Furthermore, the expression of $a(r)$ directly implies that for $B \in (0, 1/2)$,

$$a(r) \geqslant -\frac{6B}{1+r^2}\sigma^{-1} \geqslant -6B. \tag{2.12}$$

To obtain the inequality on the right-hand side of (2.11), we define two functions $a_1(r)$ and $a_2(r)$ by

$$a_1(r) = 4B(B+1)r^2 - 6B\sigma - 2Br^2\sigma$$

and

$$a_2(r) = 4B(2B-1) + 4B(2B+1-\sigma)r^2 - 4B\sigma.$$

Since

$$a(r) = \frac{\sigma^{-2}}{(1+r^2)^2} a_1(r), \qquad a_1'(r) = \frac{r}{1+r^2} a_2(r), \tag{2.13}$$

and $\sigma \geqslant 1$, for $0 < B < 1/2$, we have

$$a_2(0) = 8B(B-1) < 0$$

and

$$a_2'(r) = 8Br(2B-\sigma) \leqslant 8Br(2B-1) < 0.$$

Thus, $a_2(r) \leqslant a_2(0) < 0$ which in turn implies that $a_1'(r) < 0$. Finally, $a_1(r) \leqslant a_1(0) = -6B$ and then

$$a(r) = \frac{\sigma^{-2}}{(1+r^2)^2} a_1(r) \leqslant -\frac{6B\sigma^{-2}}{(1+r^2)^2} < 0, \tag{2.14}$$

which completes the proof of the lemma. □

3. Subsolution and supersolution

The subsolution and supersolution to the problem (2.1) can be constructed under the following assumption on the decay rate of $\eta(x)$ when x tends to infinity.

The main assumption on the global existence of solutions to the problem (1.5) and (1.6) is

(M) There exist positive constants C and B such that

$$\left|\eta(x)\right| \leqslant C\sigma^{-B}, \quad x \in \mathbf{R}^3. \tag{3.1}$$

In the following discussion, without loss of generality, we assume that $0 < B < 1/2$.

Firstly, we formally define a subsolution U by

$$U = g^{-1}\left(\eta - D_1\sigma^{-B}\left(A^2 - a(r)\right)\right) \tag{3.2}$$

and a supersolution W by

$$W = g^{-1}\left(\eta + D_2\sigma^{-B}\left(A^2 - a(r)\right)\right). \tag{3.3}$$

In the following lemmas, we will show that both U and W given in (3.2)–(3.3) are well defined. Furthermore, we will show that by appropriately choosing the parameters A, D_1 and D_2, the functions U and W satisfy the conditions (2.5) and (2.6).

Lemma 3.1. *Under the condition* (M), *if*

$$D_1 \geqslant C, \qquad A \geqslant \max\left\{\exp\left(\frac{D_1}{2}+1\right), \sqrt{6BD_1 + C}\right\}, \tag{3.4}$$

then we have:

(1) *U given in* (3.2) *is well defined so that*

$$\eta - D_1\sigma^{-B}\big(A^2 - a(r)\big) \geqslant g_{\min}. \tag{3.5}$$

(2) *$U \geqslant -\ln A^2$ $U \to 0$, as $|x| \to \infty$.*
(3) *$\mathcal{L}U = -D_1\sigma^{-B} \geqslant U$.*

Proof. Since $\eta(x) \leqslant C$ from (3.1), by using Lemma 2.4, we have

$$\begin{aligned}
&\eta - D_1\sigma^{-B}\big(A^2 - a(r)\big) - g_{\min} \\
&\quad = \eta - D_1\sigma^{-B}\big(A^2 - a(r)\big) + A^2\big(\ln A^2 - 1\big) + 1 \\
&\quad \geqslant -C - D_1\big(A^2 + 6B\big) + A^2\big(\ln A^2 - 1\big) + 1 \\
&\quad \geqslant A^2 - (6BD_1 + C) + 1 \geqslant 0,
\end{aligned}$$

where we have used

$$\ln A^2 - D_1 - 1 \geqslant 1, \qquad A^2 \geqslant 6BD_1 + C,$$

by (3.4). Hence, (3.5) holds. From Lemma 2.1, U is well defined so that $U \geqslant g^{-1}(g_{\min}) = -\ln A^2$. Since $g^{-1}(0) = 0$, we have

$$U = g^{-1}\big(\eta - D_1\sigma^{-B}\big(A^2 - a(r)\big)\big) \quad \Rightarrow \quad g^{-1}(0) = 0$$

as $|x| \to \infty$.

By the definition (3.2) for U, we have

$$\eta - g(U) = D_1\sigma^{-B}\big(A^2 - a(r)\big).$$

On the other hand, Lemma 2.4 implies that

$$\big(\Delta - A^2\big)\big(-D_1\sigma^{-B}\big) = D_1\sigma^{-B}\big(A^2 - a(r)\big).$$

Thus,

$$\big(\Delta - A^2\big)\big(-D_1\sigma^{-B}\big) = \eta - g(U).$$

By the definition of $\mathcal{L}$ and the uniqueness of the solution to (2.3), we have

$$\mathcal{L}U = -D_1\sigma^{-B}. \tag{3.6}$$

Finally, we prove $\mathcal{L}U \geqslant U$ as follows. Firstly, (3.4) implies that $\ln A^2 \geqslant D_1$ which gives

$$\mathcal{L}U = -D_1\sigma^{-B} \geqslant -D_1 \geqslant -\ln A^2.$$

Thus, $\mathcal{L}U$ and U are in the domain where g is increasing. Therefore, to show that $U \leqslant \mathcal{L}U$ is equivalent to prove

$$g(U) \leqslant g(\mathcal{L}U),$$

i.e.,

$$\eta - D_1\sigma^{-B}\big(A^2 - a(r)\big) \leqslant g\big(-D_1\sigma^{-B}\big). \tag{3.7}$$

Since Lemma 2.1 implies that

$$g\big(-D_1\sigma^{-B}\big) \geqslant \big(A^2 - 1\big)\big(-D_1\sigma^{-B}\big),$$

we have

$$
\begin{aligned}
&g\bigl(-D_1\sigma^{-B}\bigr)-\eta+D_1\sigma^{-B}\bigl(A^2-a(r)\bigr)\\
&\quad\geqslant\bigl(A^2-1\bigr)\bigl(-D_1\sigma^{-B}\bigr)-C\sigma^{-B}+D_1\sigma^{-B}\bigl(A^2-a(r)\bigr)\\
&\quad=\sigma^{-B}\bigl(D_1-C-D_1a(r)\bigr)\geqslant 0,
\end{aligned}
$$

where we have used (3.4) and Lemma 2.4 in the last inequality. Hence, (3.7) holds and this completes the proof of Lemma 3.1. □

The following lemma is about the supersolution W.

Lemma 3.2. *Under the condition* (M)*, by choosing*

$$D_2\geqslant\max\left\{C\ln 4,\frac{C}{6B}(2C)^{2/B}\exp\bigl(2(2C)^{1/B}-2\bigr)\right\},\qquad A\geqslant\max\{e,\sqrt{C}\},\tag{3.8}$$

we have:

(1) W *given in* (3.3) *is well defined so that*

$$\eta+D_2\sigma^{-B}\bigl(A^2-a(r)\bigr)\geqslant g_{\min}.\tag{3.9}$$

(2) $-\ln A^2\leqslant U\leqslant W$ *and* $W(x)\to 0$ *as* $|x|\to\infty$.
(3) $\mathcal{L}W=D_2\sigma^{-B}\leqslant W$.

Proof. Similar to the proof of (3.5), we have

$$
\begin{aligned}
&\eta+D_2\sigma^{-B}\bigl(A^2-a(r)\bigr)-g_{\min}\\
&\quad=\eta+D_2\sigma^{-B}\bigl(A^2-a(r)\bigr)+A^2\bigl(\ln A^2-1\bigr)+1\\
&\quad\geqslant A^2\bigl(\ln A^2-1\bigr)-C+1\geqslant 0,
\end{aligned}
$$

where we have used

$$\ln A^2-1\geqslant 1,\quad A^2\geqslant C,$$

from (3.8). It is then straightforward to check that $-\ln A^2\leqslant U\leqslant W$ and when $|x|\to\infty$,

$$W(x)=g^{-1}\bigl(\eta+D_2\bigl(A^2-a(r)\bigr)\bigr)\quad\Rightarrow\quad g^{-1}(0)=0.$$

To prove $\mathcal{L}W\leqslant W$, similar to (3.6), we first notice that

$$\mathcal{L}W=D_2\sigma^{-B}.\tag{3.10}$$

By the monotonicity of $\mathcal{L}$ and Lemma 3.1, we have

$$\mathcal{L}W\geqslant\mathcal{L}U\geqslant U\geqslant-\ln A^2.$$

Hence, both $\mathcal{L}W$ and W are in the domain where g is increasing. Thus, it suffices to show that

$$g(\mathcal{L}W)\leqslant g(W).\tag{3.11}$$

From (3.3) and (3.10), (3.11) is equivalent to

$$g\bigl(D_2\sigma^{-B}\bigr)\leqslant\eta+D_2\sigma^{-B}\bigl(A^2-a(r)\bigr).\tag{3.12}$$

Notice that

$$\begin{aligned}
&\eta + D_2\sigma^{-B}\big(A^2 - a(r)\big) - g\big(D_2\sigma^{-B}\big) \\
&\quad = \eta + D_2\sigma^{-B}\big(A^2 - a(r)\big) - \exp\big(-D_2\sigma^{-B}\big) - A^2 D_2\sigma^{-B} + 1 \\
&\quad \geqslant -C\sigma^{-B} + \frac{6BD_2}{(1+r^2)^2}\sigma^{-B-2} - \exp\big(-D_2\sigma^{-B}\big) + 1 \geqslant 0.
\end{aligned} \tag{3.13}$$

(3.13) gives (3.12) and then the proof of Lemma 3.2 is complete. □

Now based on the above estimates, we are ready to prove Theorem 1.1 as follows.

Proof of Theorem 1.1. Suppose that $\rho_0(x)$ satisfies (1.9) which is equivalent to the condition (M). For fixed positive constants C and B with $0 < B < 1/2$, by Lemmas 3.1 and 3.2, there exists a solution $\bar{U}(x)$ to the problem (2.1) satisfying

$$-D_1\big(1 + \ln\big(1 + |x|^2\big)\big)^{-B} \leqslant \mathcal{L}U(x) \leqslant \bar{U}(x) \leqslant \mathcal{L}W(x) = D_2\big(1 + \ln\big(1 + |x|^2\big)\big)^{-B}, \tag{3.14}$$

where positive constants D_1 and D_2 are defined in (3.4) and (3.8), respectively. Thus, $\Phi = -\bar{U}$ is a solution to the problem (1.8) and satisfies (1.10). (3.14) implies that $\bar{U}(x) \in L^1_{\mathrm{loc}}(\mathbf{R}^3)$. Moreover, since $\bar{U}(x)$ is the fixed point of the operator $\mathcal{L}$, we have

$$\bar{U}(x) = \mathcal{L}\bar{U}(x) = -\frac{1}{4\pi|x|}\exp\big(-A|x|\big) * \big(\eta - g(\bar{U})\big) \in C^\infty\big(\mathbf{R}^3\big),$$

where the positive constant A satisfies

$$A \geqslant \max\left\{\exp\left(\frac{D_1}{2} + 1\right), \sqrt{6BD_1 + C}\right\}. \tag{3.15}$$

Hence, $\Phi(x) = -\bar{U}(x)$ is smooth.

For the uniqueness, suppose Φ_1 and Φ_2 are two solutions to the problem (1.8) so that

$$\begin{cases} \Delta(\Phi_1 - \Phi_2) = \exp(\Phi_1) - \exp(\Phi_2), & x \in \mathbf{R}^3, \\ \Phi_i \to 0, \quad \text{as } |x| \to \infty,\ i = 1, 2. \end{cases}$$

Then $\Phi_1 - \Phi_2 = 0$ comes directly from the maximum principle for the elliptic equation and this completes the proof of Theorem 1.1. □

References

[1] N. Ben Abdallah, J. Dolbeault, Relative entropies for kinetic equations in bounded domain (irreversibility, stationary solution, uniqueness), Arch. Ration. Mech. Anal. 168 (2003) 253–298.

[2] L.L. Bonilla, J.A. Carrillo, J. Soler, Asymptotic behavior of the initial boundary value problem for the three-dimensional Vlasov–Poisson–Fokker–Planck system, SIAM J. Appl. Math. 57 (1997) 1343–1372.

[3] F. Bouchut, Existence and uniqueness of a global smooth solution for the Vlasov–Poisson–Fokker–Planck system in three dimensions, J. Funct. Anal. 111 (1993) 239–258.

[4] F. Bouchut, J. Dolbeault, On long time asymptotics of the Vlasov–Fokker–Planck equation and of the Vlasov–Poisson–Fokker–Planck system with Coulombic and Newtonian potentials, Differential Integral Equations 8 (1995) 487–514.

[5] L. Devilletes, J. Dolbeault, On long time asymptotics of the Vlasov–Poisson–Boltzmann equation, Comm. Partial Differential Equations 16 (1991) 451–489.

[6] J. Dolbeault, Free energy and solutions of the Vlasov–Poisson–Fokker–Planck system: External potential and confinement (large time behavior and steady states), J. Math. Pures Appl. 78 (1999) 121–157.

[7] K. Dressler, Stationary solutions of the Vlasov–Poisson–Fokker–Planck equation, Math. Methods Appl. Sci. 9 (1987) 169–176.

[8] K. Dressler, Steady states in plasma physics—The Vlasov–Fokker–Planck equation, Math. Methods Appl. Sci. 12 (1990) 471–487.
[9] R. Glassey, J. Schaeffer, Y. Zheng, Steady states of the Vlasov–Poisson–Fokker–Planck system, J. Math. Anal. Appl. 202 (1996) 1058–1075.
[10] Y. Guo, The Vlasov–Poisson–Boltzmann system near vacuum, Comm. Math. Phys. 218 (2001) 293–313.
[11] S. Mischler, On the initial boundary value problem for the Vlasov–Poisson–Boltzmann system, Comm. Math. Phys. 210 (2000) 447–466.
[12] T. Yang, H.-J. Yu, H.-J. Zhao, Cauchy problem for the Vlasov–Poisson–Boltzmann system, Arch. Ration. Mech. Anal., in press.
[13] T. Yang, H.-J. Zhao, Global existence of classical solutions to the Vlasov–Poisson–Boltzmann system, Comm. Math. Phys., in press.